PERGAMON

International Journal of Solids and Structures 37 (2000) 4635–4654

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Rigid stamp indentation for a thermoelastic half-plane

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Received 22 June 1998; in revised form 30 April 1999

Abstract

A problem for the thermoelastic half-plane indented by a rigid punch of various shapes is solved explicitly in this paper by the method of analytical continuation. The effects of applied loadings, the profile of the punch and material properties on the contact stress under the punch face are studied in detail and shown in graphic form. A rigid punch of three different profiles with or without friction is considered under the complete or incomplete indentation condition. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Rigid punch; Indentation; Contact stress

1. Introduction

Indentation of a punch into the surface of an elastic half-plane has been studied for many years because it is a fundamental model for explaining forming and contact damage processes. This is one of the mixed boundary value problems where both the stress and the displacement are prescribed on the half-plane surface. One of the most powerful methods for solving plane punch problems is based on complex variable theory with the aid of the technique of analytical continuation. Following the method of analytical continuation, it is convenient to reduce the punch problems to the problem of linear relationship where the general solution can be expressed in terms of the basic Plemelj function. Using this method Muskhelishvili (1953) and England (1971) provided solutions for several types of punch problems in their books. By combining Stroh's formalism (Stroh, 1958) and continuation theorem, Fan and Hwu (1996) solved punch problems for an anisotropic elastic half-plane. Another powerful mathematical tool for solving punch problems is based on the integral transforms that the problem is formulated in terms of integral equations. The inverse of the integral equations can be obtained by the use of some orthogonal polynomial expansions which was summarized by Gladwell and England (1977)

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0020-7683/00/$ - see front matter
PII: S0020-7683(99)00138-9

and Gladwell (1978). The problem for a punch of arbitrary shape on an elastic half-plane was considered by Fabrikant (1986a, b) who employed an integral equation based on the reciprocal distance established by himself. Shibuya et al. (1989) used the generalized Abel transform method to study a frictional punch problem of elliptical shape on an elastic half-plane. Willis (1966) investigated the Hertzian contact problem of anisotropic bodies based on the method of Fourier transform.

The research noted above has considered only the isothermal case. When heat flows between two conducting solids, there will be some resistance to heat flow across the interface and the contact pressure will be influenced by the temperature field in the bodies. In certain cases, separation will occur at the corners of the punch and results in incomplete indentation which makes the problem more complicated. The literature on this subject includes works by Keer and Fu (1967), Fu (1970), Barber (1973, 1978), Clements and Toy (1976), Gladwell and Barber (1983), Chao et al. (1999). More specifically, for the problem with a frictionless rigid flat-ended punch, Comninou et al. (1981) found that, depending on the magnitude and the direction of the total heat flux, either perfect thermal contact throughout the punch face or an imperfect contact region at the center with adjacent perfect contact regions occurs. In the present paper, the thermoelastic problem of indentation of plane punches with various profiles into an elastic half-plane is solved by the method of analytical continuation. The frictional punch problems are also studied in this paper. Examples of incomplete indentation by a flat-ended, wedge-ended or parabolic-ended punch are solved explicitly and the condition to have complete indentation is discussed.

2. Heat conduction through a punch

The relationship of the heat fluxes and the temperature gradient in an isotropic plane medium can be expressed as

$$h_i = -kT_{,i} \quad (i = 1, 2) \tag{1}$$

where k, T, h_1, h_2 stand for the thermal conductivity, temperature and heat fluxes in the x_1, x_2 directions, respectively. A subscript after a comma stands for a differentiation with respect to the associated index. For the steady-state heat conduction problem, the temperature function satisfying the Laplace's equation can be expressed in terms of a single potential function as

$$T(z) = \mathrm{Re}[\phi_0(z)] \tag{2}$$

where Re denotes the real part of a complex function. Now we consider the case that a rigid punch of width $2a$ is pressed into the half-plane $x_2 > 0$ by a total force $X + iY$ and a total heat flux Q from the punch to the half plane as indicated in Fig. 1. If there is perfect thermal contact throughout the contact region $-c_2 < x_1 < c_1$ pressed by the punch and the remaining region of the half plane is assumed to be thermally insulated (Fig. 1), these conditions give

$$\frac{\mathrm{d}T(x_1, 0)}{\mathrm{d}x_1} = 0, \qquad -c_2 < x_1 < c_1 \tag{3}$$

and

$$h_2(x_1, 0) = 0, \qquad x_1 > c_1 \quad \text{or} \quad x_1 < -c_2. \tag{4}$$

Eqs. (1)–(4) lead to the following Hilbert problem

$$\Phi_0^+(x_1) + \Phi_0^-(x_1) = 0, \qquad -c_2 < x_1 < c_1 \tag{5}$$

with

$$\Phi_0(z) = \phi_0'(z). \tag{6}$$

The superscript + (or −) is used to denote the field quantities approached from the medium $x_2 > 0$ (or $x_2 < 0$) and the overbar stands for the complex conjugate.

The solution to the Hilbert problem can be obtained as (Muskhelishvili, 1953)

$$\Phi_0(z) = \frac{b_1 z + b_2}{\sqrt{(z + c_2)(z - c_1)}}, \quad z \in S^+. \tag{7}$$

The constant b_1 in (7) vanishes due to the fact that the heat flux tends to be zero at infinity and the remaining constant b_2 can be determined from

$$\int_{-c_2}^{c_1} h_2 \, dx_1 = Q. \tag{8}$$

It yields

$$b_2 = -\frac{Q}{k\pi}. \tag{9}$$

Therefore, the final solution for $\Phi_0(z)$ becomes

$$\Phi_0(z) = \frac{-Q}{k\pi\sqrt{(z + c_2)(z - c_1)}}, \quad z \in S^+. \tag{10}$$

Clearly (10) may be integrated to give

$$\phi_0(z) = \int \Phi_0(z) \, dz = -\frac{Q}{ik\pi} \sin^{-1}\frac{2z - c_1 + c_2}{c_1 + c_2}. \tag{11}$$

When the property of symmetry with the contact length $c_1 = c_2 = c$ is satisfied, the solution given by (11)

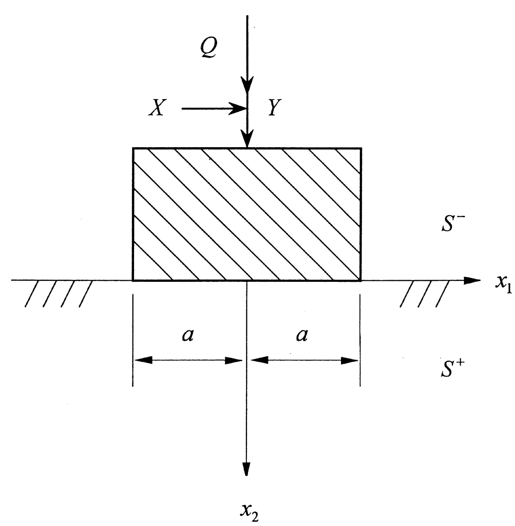

Fig. 1. Geometry of the flat-ended punch.

becomes

$$\phi_0(z) = \int \Phi_0(z)\,\mathrm{d}z = -\frac{Q}{ik\pi}\sin^{-1}\frac{z}{c}. \tag{12}$$

3. Thermoelastic field for punch problems

For the two-dimensional theory of thermoelasticity, the components of the displacement, resultant force and the components of the stress can be represented in terms of two stress potentials $\phi(z)$, $\psi(z)$ and a temperature potential $\phi_0(z)$ as (Bogdanoff, 1954)

$$2\mu(u+iv) = \kappa\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)} + 2\mu\beta\int\phi_0(z)\,\mathrm{d}z$$

$$-Y + iX = \phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)}$$

$$\sigma_{11} + \sigma_{22} = 2[\phi'(z) + \overline{\phi'(z)}]$$

$$\sigma_{22} - i\sigma_{12} = \phi'(z) + \overline{\phi'(z)} + z\overline{\phi''(z)} + \overline{\psi'(z)} \tag{13}$$

where the symbol μ denotes the shear modulus and $\kappa=(3-4\nu)/(1+\nu)$, $\beta=\alpha$ for plane stress and $\kappa=3-4\nu$, $\beta=(1+\nu)\alpha$ for plane strain with ν being the Poisson's ratio and α the thermal expansion coefficient. When the surface of the half-plane is pressed by a rigid punch of width $2a$, the stresses occur at the contact region and the remaining region of the half-plane surface is unstressed. The unstressed region may be regarded as the traction free boundary and from (13) we have

$$\phi(x_1) + x_1\overline{\phi'(x_1)} + \overline{\psi(x_1)} = 0, \qquad |x_1| > a,\ x_2 = 0. \tag{14}$$

Using the continuation theorem, the boundary condition (14) allows us to extend the definition of $\phi(z)$ from S^+ into S^- by putting

$$\phi(z) = \phi(z), \quad z \in S^+$$

$$\phi(z) = -z\overline{\phi'(\bar{z})} - \overline{\psi(\bar{z})}, \quad z \in S^-. \tag{15}$$

Hence $\phi(z)$ is continued analytically from S^+ into S^-, and holomorphic in the whole plane. (15) may be rearranged to express $\psi(z)$ in terms of $\phi(z)$ as

$$\psi(z) = -\overline{\phi(\bar{z})} - z\phi'(z), \quad z \in S^+. \tag{16}$$

Using (16), (13) can be modified to the following expressions:

$$2\mu(u+iv) = \kappa\phi(z) + \phi(\bar{z}) + (\bar{z}-z)\overline{\phi'(z)} + 2\mu\beta\int\phi_0(z)\,\mathrm{d}z$$

$$-Y + iX = \phi(z) - \phi(\bar{z}) - (\bar{z}-z)\overline{\phi'(z)}$$

$$\sigma_{11} + \sigma_{22} = 2[\Phi(z) + \overline{\Phi(z)}]$$

$$\sigma_{22} - i\sigma_{12} = \Phi(z) - \Phi(\bar{z}) + (z - \bar{z})\overline{\Phi'(z)} \tag{17}$$

where

$$\Phi(z) = \phi'(z). \tag{18}$$

Now, a complete solution to the half-plane problem has been reduced to the evaluation of a single complex function $\phi(z)$ (or $\Phi(z)$) which must satisfy the prescribed boundary conditions along the half plane surface.

4. Frictionless punch problems

In this section we consider the case of a rigid frictionless punch indenting the half-plane S^+ under the action of a total force Y and a heat flux Q. The boundary conditions for this problem can be expressed as

$$\left.\begin{array}{l} v = f(x_1) + \text{constant} \\ \sigma_{12} = 0 \end{array}\right\} \quad \text{on } x_2 = 0, \quad |\, x_1 \,| \le a. \tag{19}$$

From (17), the condition $\sigma_{xy} = 0$ on $x_2 \to 0^+$, $|x_1| \le a$ implies

$$\sigma_{12} = -\frac{1}{2i}\{\Phi^+(x_1) - \overline{\Phi^+(x_1)} - \Phi^-(x_1) + \overline{\Phi^-(x_1)}\} = 0 \tag{20}$$

(20) allows us to introduce a new holomorphic function defined as

$$\theta(z) = \begin{cases} \Phi(z) + \overline{\Phi(\bar{z})}, & z \in S^+ \\ \Phi(z) + \overline{\Phi(\bar{z})}, & z \in S^- \end{cases}. \tag{21}$$

Thus the function $\Phi(z) + \overline{\Phi(\bar{z})}$ is holomorphic in the whole plane and by Liouville's theorem is equal to zero, and for $x_2 \to 0^+$, we have

$$\Phi^+(x_1) + \overline{\Phi^-(x_1)} = 0. \tag{22}$$

With the aid of (19) and (22), (17) for $x_2 \to 0^+$ in differentiated form yields

$$\Phi^+(x_1) + \Phi^-(x_1) = \frac{4\mu}{1+\kappa}[if'(x_1) - \beta\phi_0^+(x_1)]. \tag{23}$$

The general function $\Phi(z)$ satisfying (23) has the form

$$\Phi(z) = \frac{X(z)}{2\pi i}\int_{-a}^{a} \frac{\frac{4\mu}{1+\kappa}[if'(t) - \beta\phi_0^+(t)]}{(t-z)X^+(t)}\mathrm{d}t + FX(z) \tag{24}$$

where the Plemelj function $X(z)$ is defined as

$$X(z) = (z^2 - a^2)^{-1/2}.$$

The constant F in (24) can be determined from the condition

$$-Y=\int_{-a}^{a}\sigma_{22}\,\mathrm{d}t. \tag{25}$$

Substituting (24) into (17) and using (25), we obtain

$$F=-\frac{iY}{2\pi}. \tag{26}$$

With this result, $\Phi(z)$ in (24) becomes

$$\Phi(z)=\frac{X(z)}{2\pi i}\int_{-a}^{a}\frac{\frac{4\mu}{1+\kappa}[if'(t)-\beta\phi_0^{+}(t)]}{(t-z)X^{+}(t)}\mathrm{d}t+\left(-\frac{iY}{2\pi}\right)X(z). \tag{27}$$

The normal stress under the punch can be obtained from (17) as

$$\sigma_{22}=\frac{1}{\pi i}X^{+}(x_1)\int_{-a}^{a}\frac{\frac{4\mu}{1+\kappa}[if'(t)-\beta\phi_0^{+}(t)]}{(t-z)X^{+}(t)}\mathrm{d}t-\frac{iY}{\pi}X^{+}(x_1). \tag{28}$$

The problem is explicitly solved. Now we consider that two distinct situations can occur: perfect contact throughout the punch face, and separation at the punch corners, with three different profiles of the punch.

4.1. Flat-ended punch

As our first example the punch of width $2a$ is flat-ended then $f'(x_1)=0$ and from (12), the normal stress under the punch in (28) becomes

$$\sigma_{22}=\frac{1}{\sqrt{a^2-x_1^2}}\left[-\frac{Y}{\pi}-\frac{4\mu\beta Q}{\pi^2k(1+\kappa)}\int_{-a}^{a}\frac{\sqrt{a^2-t^2}}{t-x_1}\sin^{-1}\frac{t}{a}\,\mathrm{d}t\right]. \tag{29}$$

The integral in (29) cannot be directly evaluated in closed form. However, it is an even function of x_1 varying monotonically from $x_1=0$ to $x_1=a$, the extremal values being

$$\int_{-a}^{a}\frac{\sqrt{a^2-t^2}}{t}\sin^{-1}\frac{t}{a}\,\mathrm{d}t=2a(2G-1) \tag{30}$$

and

$$\int_{-a}^{a}\frac{\sqrt{a^2-t^2}}{t-a}\sin^{-1}\frac{t}{a}\,\mathrm{d}t=-2a \tag{31}$$

where G = 0.915965 ... is Catalan's constant (Gradshteyn and Ryzhik, 1965). (29) will define a compressive stress throughout $|x_1|<a$ if and only if

$$-1.89<\lambda^*<1.57$$

where

$$\lambda^* = \frac{4\mu\beta Qa}{k(1+\kappa)Y}. \tag{32}$$

For $\lambda^* > 1.57$, separation begins to occur at the punch corners and leaves a region of perfect contact $|x_1| < c < a$. Note that the normal traction must be a negative value throughout the contact region and the related contact length $2c$ can be determined from the continuity condition at the transition such that the function, (29), must be bounded at $x_1 = \pm c$. The final result for the contact length is found to be

$$\frac{c}{a} = \frac{\pi}{2\lambda^*}. \tag{33}$$

For separation, the normal stress under the punch is

$$\sigma_{22} = \frac{1}{\sqrt{c^2 - x_1^2}}\left[-\frac{Y}{\pi} - \frac{4\mu\beta Q}{\pi^2 k(1+\kappa)}\int_{-c}^{c} \frac{\sqrt{c^2 - t^2}}{t - x_1} \sin^{-1}\frac{t}{c}\,\mathrm{d}t \right]. \tag{34}$$

Fig. 2 shows the normal stress distribution for $-1.89 < \lambda^* < 1.57$ where perfect contact is maintained throughout the punch face and there is a square root singularity at the edge of the punch except for the transitional case $\lambda^* = 1.57$. For $\lambda^* > 1.57$, separation occurs at the corners of the punch, leaving a contact width $2c$ which varies inversely with λ^* as indicated in (33). Typical contact pressure distributions for $\lambda^* > 1.57$ are shown in Fig. 3. For $\lambda^* < -1.89$, an imperfect contact region at the center with adjacent perfect contact regions occurs (Comninou et al., 1981). We do not further discuss this case throughout this study.

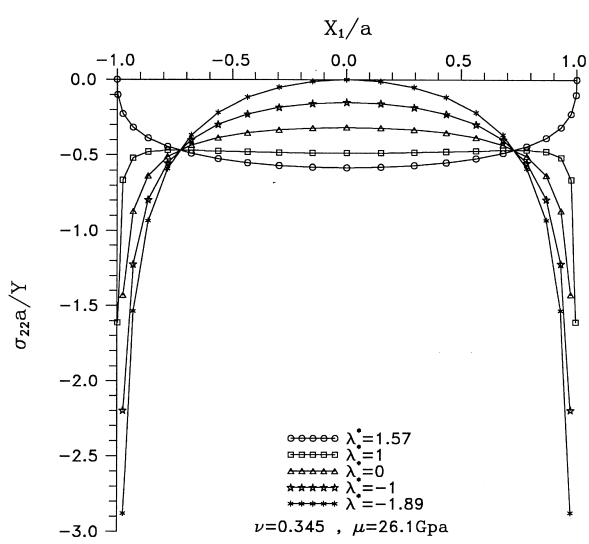

Fig. 2. Contact pressure distribution for perfect contact of smooth flat punch.

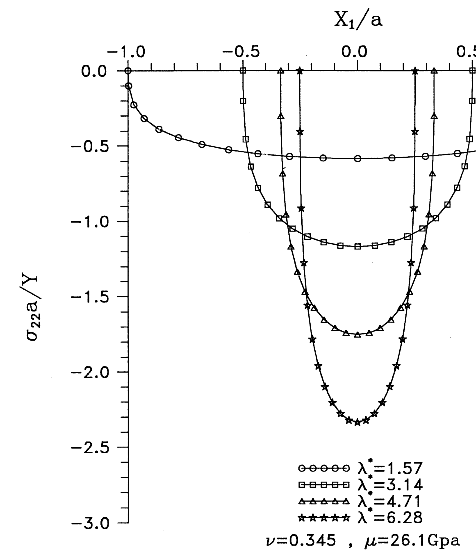

Fig. 3. Contact pressure distribution for separation of smooth flat punch.

4.2. Wedge-ended punch

As a second example we consider the motion to the right of the wedge-shaped punch shown in Fig. 4 where the end section of the punch is $f'(x_1) = \varepsilon$ and we take the origin so that the contact region is $-a < x_1 < a$. The integral term in (28) may be evaluated to find the stress distribution under the punch being

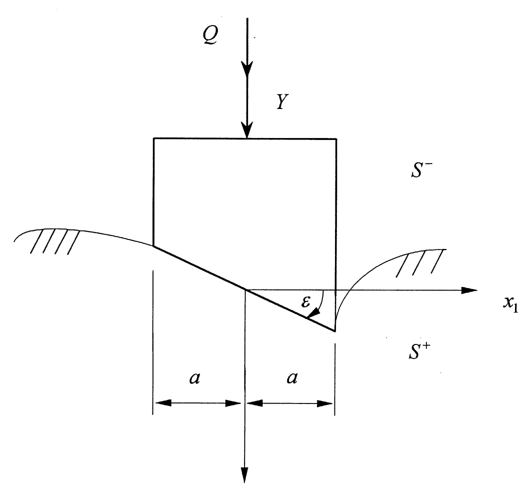

Fig. 4. Geometry of the wedge-ended punch.

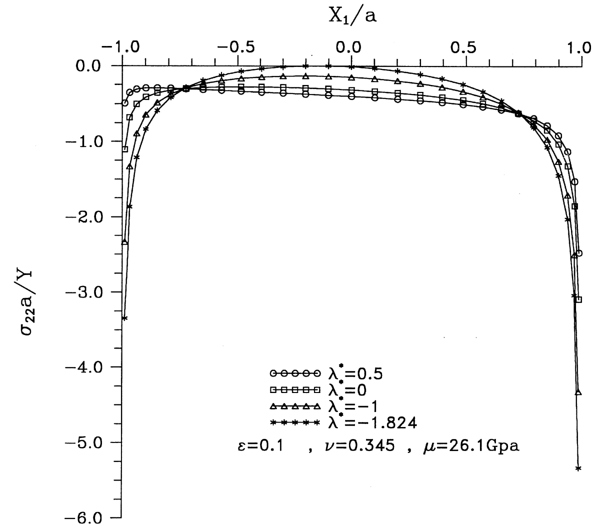

Fig. 5. Contact pressure distribution for perfect contact of smooth wedge punch.

$$\sigma_{22} = \frac{1}{\sqrt{a^2 - x_1^2}} \left[-\frac{Y}{\pi} - \frac{4\mu\varepsilon x_1}{1+\kappa} - \frac{4\mu\beta Q}{\pi^2 k(1+\kappa)} \int_{-a}^{a} \frac{\sqrt{a^2 - t^2}}{t - x_1} \sin^{-1}\frac{t}{a}\, \mathrm{d}t \right]. \tag{35}$$

In view of (35), the range of λ^* defined as (32), which is dependent of the material properties μ and κ, and the slope of the end section of the punch ε, for compressive contact stress throughout the contact region can not be obtained explicitly. If the half-plane is made of aluminum with $\mu = 26.1$ GPa, $\nu = 0.345$ and the slope is taken to be $\varepsilon = 0.1$, the normal stress distribution for $-1.824 < \lambda^* < 0.823$, which ensures perfect contact throughout the punch face, is displayed in Fig. 5. For $\lambda^* > 0.823$, separation

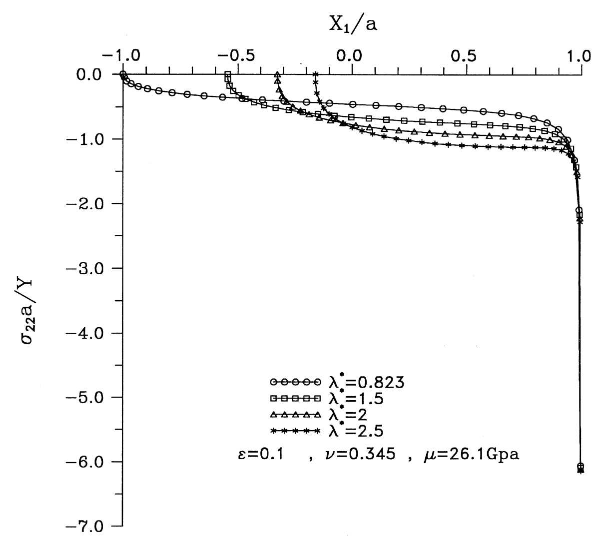

Fig. 6. Contact pressure distribution for separation of smooth wedge punch.

begins to occur at the left corner of the punch where the length of the contact region can be determined from the condition that the contact stress must be bounded at $x_1 = -c_2$. Applying (35) and using the extremal values in (30) and (31), we obtain the following explicit relation

$$c_2 = \frac{\dfrac{Y}{\pi} - \dfrac{\lambda^* Y}{\pi^2}}{\dfrac{4\mu\varepsilon}{1+\kappa} + \dfrac{\lambda^* Y}{a\pi^2}}. \tag{36}$$

By setting $\lambda^* = 0$, (36) reduces to the result of the corresponding isothermal problem given by England (1971). For separation, the normal stress under the punch is

$$\sigma_{22} = \frac{1}{\sqrt{(a-x_1)(x_1+c_2)}}\left[-\frac{Y}{\pi} - \frac{4\mu\varepsilon x_1}{1+\kappa} - \frac{4\mu\beta Q}{\pi^2 k(1+\kappa)}\int_{-c_2}^{a} \frac{\sqrt{(a-t)(t+c_2)}}{t-x_1} \sin^{-1}\frac{2t-a+c_2}{a+c_2}\,\mathrm{d}t\right]. \tag{37}$$

Numerical results of the stress distribution for $\lambda^* > 0.823$ are depicted in Fig. 6. If we consider the motion to the left of the punch, the contact length c_1, when separation occurs, can be also determined from (35). The result is

$$c_1 = \frac{-\dfrac{Y}{\pi} + \dfrac{\lambda^* Y}{\pi^2}}{\dfrac{4\mu\varepsilon}{1+\kappa} - \dfrac{\lambda^* Y}{a\pi^2}}. \tag{38}$$

where ε is a negative number.

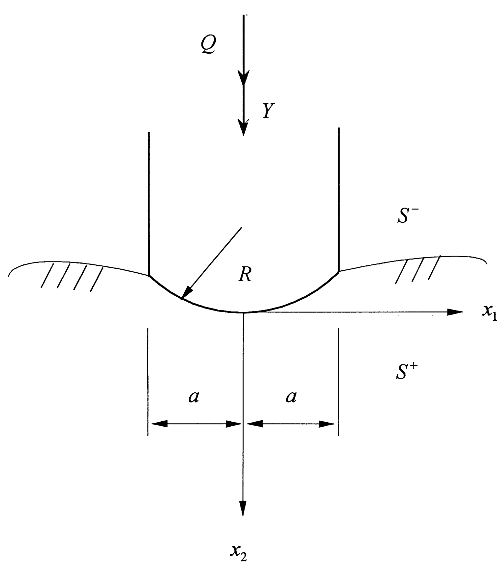

Fig. 7. Geometry of the parabolic-ended punch.

4.3. Parabolic-ended punch

As our third example we consider indentation by a parabolic punch having an end-face $f'(x_1) = -x_1/R$ (see Fig. 7). The integral term in (28) may be evaluated to find the stress distribution under the punch being

$$\sigma_{22} = \frac{1}{\sqrt{a^2 - x_1^2}}\left[-\frac{Y}{\pi} + \frac{2\mu(2x_1^2 - a^2)}{R(1+\kappa)} - \frac{4\mu\beta Q}{\pi^2 k(1+\kappa)}\int_{-a}^{a} \frac{\sqrt{a^2 - t^2}}{t - x_1} \sin^{-1}\frac{t}{a}\, \mathrm{d}t \right]. \tag{39}$$

To have a complete indentation, the contact stress in (39) must be a negative value. By the use of the extremal values in (30) and (31), (39) defines a compressive stress throughout $|x_1| < a$ if and only if

$$\frac{(1+\kappa)Y}{2\mu a^2 \pi}\left[-1 - \lambda^* \frac{2(2G-1)}{\pi} \right] < \frac{1}{R} < \frac{(1+\kappa)Y}{2\mu a^2 \pi}\left[1 - \lambda^* \frac{2}{\pi} \right]. \tag{40}$$

It is seen that the range of λ^* defined as (32) is dependent of the material properties and a radius of curvature. For the given material properties $\mu = 26.1$ GPa, $\nu = 0.345$ and a radius of curvature is taken to be $R/a = 100$, the range of λ^* is found to be $-1.933 < \lambda^* < 1.533$ for which a compressive contact stress throughout the punch face is preserved. The normal stress under the punch carried out for different values of λ^* is displayed in Fig. 8. For $\lambda^* > 1.533$, separation begins to occur at the corners of the punch, leaving a region of perfect contact $|x_1| < c < a$, where the contact length c may be determined from the condition that the contact stress (39) must be bounded at $x_1 = \pm c$. The result is

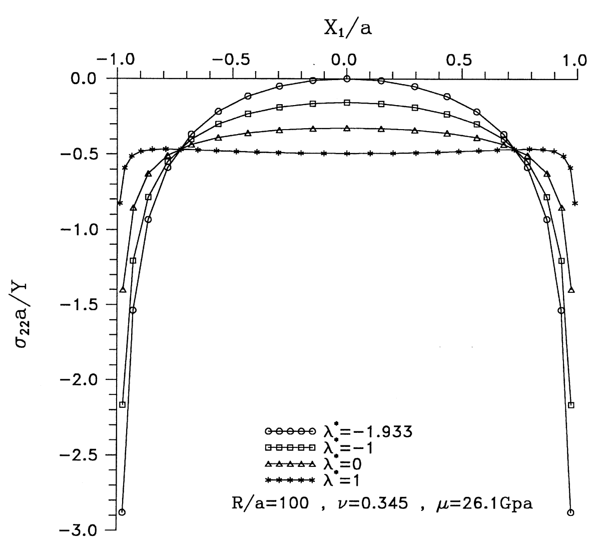

Fig. 8. Contact pressure distribution for perfect contact of smooth parabolic punch.

$$c=\frac{-\dfrac{2\lambda^* Y}{a\pi^2}+\sqrt{\left(\dfrac{2\lambda^* Y}{a\pi^2}\right)^2+\dfrac{8\mu Y}{R(1+\kappa)\pi}}}{\dfrac{4\mu}{R(1+\kappa)}}. \tag{41}$$

For the special case of $\lambda^*=0$, (41) agrees with the one of the corresponding isothermal problem provided by England (1971).

For separation, the normal stress under the punch is

$$\sigma_{22}=\frac{1}{\sqrt{c^2-x_1^2}}\left[-\frac{Y}{\pi}+\frac{2\mu(2x_1^2-c^2)}{R(1+\kappa)}-\frac{4\mu\beta Q}{\pi^2 k(1+\kappa)}\int_{-c}^{c}\frac{\sqrt{c^2-t^2}}{t-x_1}\sin^{-1}\frac{t}{c}\,\mathrm{d}t\right]. \tag{42}$$

Numerical results of the normal stress distribution, when separation occurs, are shown in Fig. 9.

5. Frictional punch problems

In this section we consider the case that friction exists between a rigid punch and the surface of a half-plane. The boundary conditions for this kind of problem may be expressed as

$$\left.\begin{array}{l} v=f(x_1)+\text{constant} \\ \sigma_{12}=\sigma_{22}\lambda \end{array}\right\} \quad \text{on } x_2=0, \qquad |x_1|\le a \tag{43}$$

where $f(x_1)$ is a given function for the profile of the punch, and λ is the friction coefficient. Note that the second equation of (43) only holds provided σ_{22} is negative, which must be checked when the solution is obtained.

By using (17), the boundary condition $\sigma_{12}=\sigma_{22}\lambda$ on the contact region implies

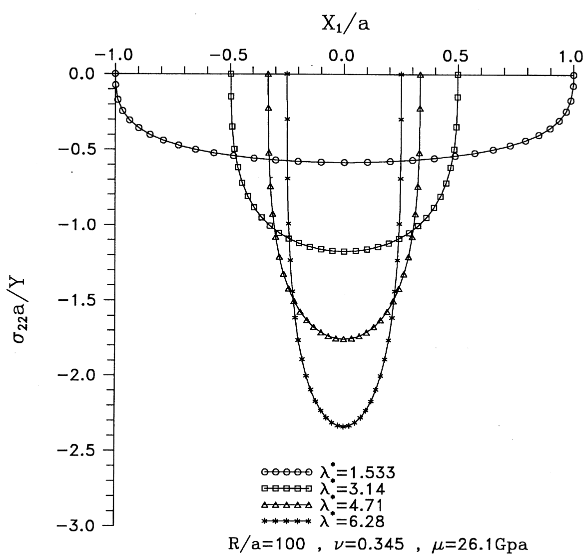

Fig. 9. Contact pressure distribution for separation of smooth parabolic punch.

$$i\{\phi'^{+}(x_1)-\overline{\phi'^{+}(x_1)}-\phi'^{-}(x_1)+\overline{\phi'^{-}(x_1)}\}=\lambda\{\phi'^{+}(x_1)+\overline{\phi'^{+}(x_1)}-\phi'^{-}(x_1)-\overline{\phi'^{-}(x_1)}\}. \tag{44}$$

Based on the method of analytical continuation, (44) permits us to introduce a new holomorphic function defined as

$$\theta(z)=\begin{cases}\phi'(z)+\dfrac{i+\lambda}{i-\lambda}\overline{\phi'(\bar{z})}, & z\in S^{+}\\[2mm] \phi'(z)+\dfrac{i+\lambda}{i-\lambda}\overline{\phi'(\bar{z})}, & z\in S^{-}\end{cases}. \tag{45}$$

Now the function

$$\phi'(z)+\frac{i+\lambda}{i-\lambda}\overline{\phi'(\bar{z})}$$

is holomorphic in the whole plane including the point at infinity, by Liouville's theorem we have

$$\phi'(z)+\frac{i+\lambda}{i-\lambda}\overline{\phi'(\bar{z})}=0$$

and for $x_2\to 0^{+}$, resulting in

$$\phi'^{+}(x_1)+\frac{i+\lambda}{i-\lambda}\overline{\phi'^{-}(x_1)}=0. \tag{46}$$

With the aid of (43) and (46), (17) for $x_2\to 0^{+}$ in differentiated form gives

$$\Phi^{+}(x_1)+m\Phi^{-}(x_1)=n^{+}(x_1) \tag{47}$$

where

$$m=\frac{\kappa(i-\lambda)+(i+\lambda)}{\kappa(i+\lambda)+(i-\lambda)}=e^{2\pi i\alpha_1}$$

$$n^{+}(x_1)=\frac{(-1+i\lambda)\cos\ \pi\alpha_1}{(\kappa+1)\mathrm{e}^{-\pi i\alpha_1}}[4\mu\beta\phi_0^{+}(x_1)-4\mu i f'(x_1)]$$

and

$$\tan\ \pi\alpha_1=\lambda\frac{\kappa-1}{\kappa+1},\quad 0\le\alpha_1<\frac{1}{2}.$$

The general solution to the Hilbert problem, (47), can be obtained as

$$\Phi(z)=\frac{X(z)}{2\pi i}\int_{-a}^{a}\frac{n^{+}(t)}{(t-z)X^{+}(t)}\,\mathrm{d}t+FX(z) \tag{48}$$

where

$$X(z)=(z+a)^{-\gamma}(z-a)^{\gamma-1},\quad \gamma=\frac{1}{2}+\alpha_1$$

The constant F appearing in (48) can be found from the condition

$$-Y = \int_{-a}^{a} \sigma_{22}\, \mathrm{d}t \tag{49}$$

and has the form

$$F = -\frac{Y(i+\lambda)}{2\pi}. \tag{50}$$

By substituting (50) into (48), the final result for $\Phi(z)$ becomes

$$\Phi(z) = \frac{2\mu}{\pi i}\frac{(-1+i\lambda)\cos\ \pi\alpha_1}{(\kappa+1)\mathrm{e}^{-\pi i\alpha_1}}X(z)\int_{-a}^{a}\frac{[\beta\phi_0^+(t) - if'(t)]}{(t-z)X^+(t)}\,\mathrm{d}t + \left(-\frac{(i+\lambda)Y}{2\pi}\right)X(z). \tag{51}$$

Substitution of (51) into (17) results in

$$\sigma_{22} - i\sigma_{12} = \Phi^+(x_1) - \Phi^-(x_1) = \frac{2\mu}{\pi i}\frac{(-1+i\lambda)}{(1+\kappa)\mathrm{e}^{-i\pi\alpha_1}}(1+\mathrm{e}^{-i2\pi\alpha_1})X^+(x_1)\int_{-a}^{a}\frac{[\beta\phi_0(t) - if'(t)]}{(t-x_1)X^+(t)}\,\mathrm{d}t - \frac{(i+\lambda)Y}{2\pi}(1+\mathrm{e}^{-i2\pi\alpha_1})X^+(x_1). \tag{52}$$

The problem now is solved in principle. For the purpose of illustration, we examine the following examples with three different profiles of the punch.

5.1. Flat-ended punch

We first consider the case of a flat-ended punch which is pressed to the right and has a contact region $|x_1| \leq a$. By using (12) and knowing that $f'(x_1) = 0$, (52) defines the normal stress under the punch as

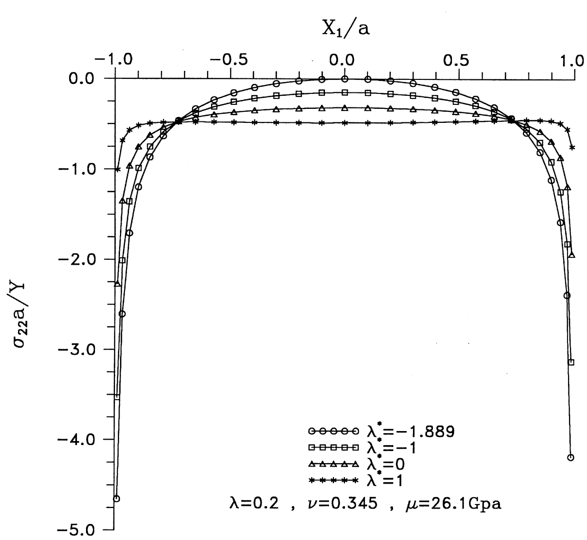

Fig. 10. Contact pressure distribution for perfect contact of frictional flat punch.

$$\sigma_{22}=\frac{1}{(x_1+a)^{\frac{1}{2}+\alpha_1}(a-x_1)^{\frac{1}{2}-\alpha_1}}\left\{-\frac{Y}{\pi}\cos\pi\alpha_1-\frac{4\mu\beta Q\ \cos^2\pi\alpha_1}{\pi^2k(1+\kappa)}\int_{-a}^{a}\frac{(t+a)^{\frac{1}{2}+\alpha_1}(a-t)^{\frac{1}{2}-\alpha_1}}{t-x_1}\sin^{-1}\frac{t}{a}\,dt\right\}. \tag{53}$$

Obviously, by letting $\alpha_1=0$, (53) is simplified to (29) for the punch problem without friction. In view of (53), the range of λ^* as defined by (32) for which a compressive contact stress throughout the contact region must be preserved can be determined numerically once the material properties of the half-plane and the slope of the punch profile are selected. It is seen that, to have a complete indentation, the range $-1.889 < \lambda^* < 1.480$ is obtained as the material constants $\mu=26.1$ GPa, $\nu=0.345$ are used and the friction coefficient is taken to be $\lambda=0.2$. Fig. 10 illustrates the normal stress distribution for different numbers of λ^* where a negative stress prevails within the contact region. As $\lambda^* > 1.480$, separation begins to occur at the right corner of the punch and the normal stress under the punch becomes

$$\sigma_{22}=\frac{1}{(x_1+a)^{\frac{1}{2}+\alpha_1}(c_1-x_1)^{\frac{1}{2}-\alpha_1}}\left\{-\frac{Y}{\pi}\cos\ \pi\alpha_1-\frac{4\mu\beta Q\ \cos^2\pi\alpha_1}{\pi^2k(1+\kappa)}\int_{-a}^{c_1}\frac{(t+a)^{\frac{1}{2}+\alpha_1}(c_1-t)^{\frac{1}{2}-\alpha_1}}{t-x_1}\sin^{-1}\frac{2t-c_1+a}{a+c_1}\,dt\right\}. \tag{54}$$

The length of the contact region c_1 is determined from the condition that the normal stress must be bounded at the point $x_1=c_1$. Typical contact stress distributions for different lengths of the contact region are shown in Fig. 11.

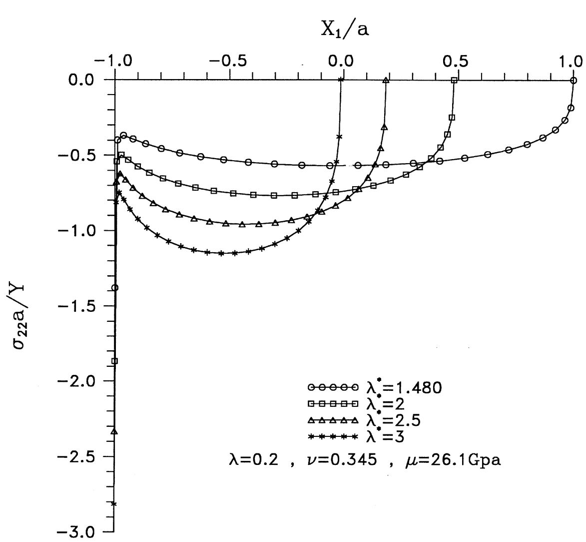

Fig. 11. Contact pressure distribution for separation of frictional flat punch.

5.2. Wedge-ended punch

As a second example we consider a wedge-ended punch which is pressed to the right and has a contact region $|x_1| \leq a$. By using (12) and knowing that $f'(x_1) = \varepsilon$, the normal stress under the punch can be obtained from (52) as

$$\sigma_{22} = \frac{1}{(x_1+a)^{\frac{1}{2}+\alpha_1}(a-x_1)^{\frac{1}{2}-\alpha_1}} \left\{ -\frac{Y}{\pi}\cos\ \pi\alpha_1 - \frac{4\mu\varepsilon(x_1+2\alpha_1 a)\cos\ \pi\alpha_1}{1+\kappa} - \frac{4\mu\beta Q\ \cos^2\pi\alpha_1}{\pi^2 k(1+\kappa)} \int_{-a}^{a} \frac{(t+a)^{\frac{1}{2}+\alpha_1}(a-t)^{\frac{1}{2}-\alpha_1}}{t-x_1} \sin^{-1}\frac{t}{a}\,\mathrm{d}t \right\}. \tag{55}$$

For a special case of $\alpha_1 = 0$, (55) is simplified to (35) for the frictionless punch problem. With similar reason as the previous approach, the range $-1.861 < \lambda^* < 0.895$ is obtained to have a complete indentation for the wedge-ended punch with the profile $f'(x_1) = \varepsilon = 0.1$. Numerical results of contact stress distributions are shown nondimensionally in Fig. 12. For $\lambda^* > 0.895$, separation occurs at the left corner of the punch leaving a contact width $a + c_2$ for which the normal stress under the contact region becomes

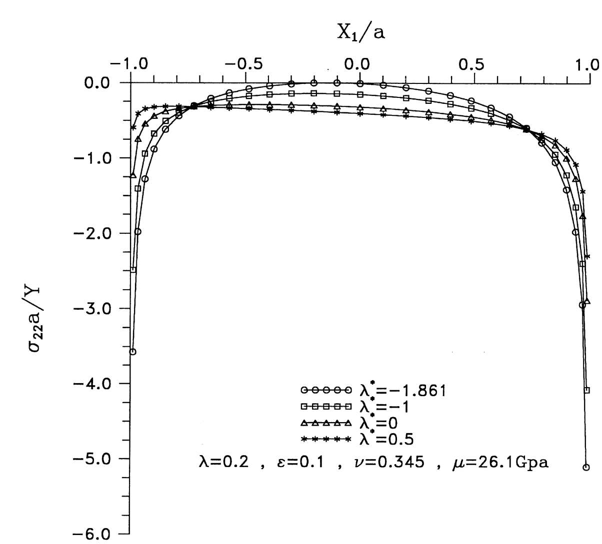

Fig. 12. Contact pressure distribution for perfect contact of frictional wedge punch.

$$\sigma_{22} = \frac{1}{(x_1+c_2)^{\frac{1}{2}+\alpha_1}(a-x_1)^{\frac{1}{2}-\alpha_1}} \Bigg\{ -\frac{y}{\pi}\cos \pi\alpha_1 - \frac{4\mu\varepsilon\left[x_1+(a+c_2)\left(\frac{1}{2}+\alpha_1\right)-a\right]\cos\ \pi\alpha_1}{1+\kappa} - \frac{4\mu\beta Q\ \cos^2\ \pi\alpha_1}{\pi^2 k(1+\kappa)}\int_{-c_2}^{a}\frac{(t+c_2)^{\frac{1}{2}+\alpha_1}(a-t)^{\frac{1}{2}-\alpha_1}}{t-x_1}\sin^{-1}\frac{2t-a+c_2}{a+c_2}\,\mathrm{d}t\Bigg\}. \tag{56}$$

The length c_2 is determined from the condition that the stress (56) must be bounded at $x_1=-c_2$ where the punch and the half-plane meet smoothly. This yields

$$\frac{Y}{\pi} + \frac{4\mu\varepsilon(a+c_2)\left(-\frac{1}{2}+\alpha_1\right)}{1+\kappa} + \lambda^*\frac{Y\ \cos\ \pi\alpha_1}{a\pi^2}\int_{-c_2}^{a}\frac{(t+c_2)^{\frac{1}{2}+\alpha_1}(a-t)^{\frac{1}{2}-\alpha_1}}{t+c_2}\sin^{-1}\frac{2t-a+c_2}{a+c_2}\,\mathrm{d}t = 0. \tag{57}$$

Numerical results of the normal stress distribution for different values of the contact length c_2 are shown in Fig. 13.

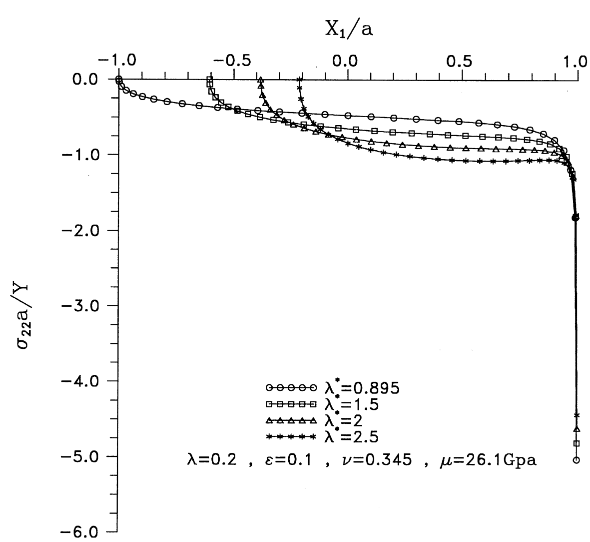

Fig. 13. Contact pressure distribution for separation of frictional wedge punch.

5.3. Parabolic-ended punch

The last example we consider indentation by a parabolic-ended punch which is pressed to the right and has a contact region $|x_1| \leq a$. By using the equation $f'(x_1) = -x_1/R$ and the temperature function (12), the normal stress under the punch can be obtained from (52) as

$$\sigma_{22} = \frac{1}{(x_1+a)^{\frac{1}{2}+\alpha_1}(a-x_1)^{\frac{1}{2}-\alpha_1}}\left\{-\frac{Y}{\pi}\cos \pi\alpha_1 + \frac{4\mu \cos \pi\alpha_1}{R(1+\kappa)}\left[x_1^2 + x_1(2a\alpha_1) + 2a^2\left(\alpha_1^2 - \frac{1}{4}\right)\right] - \frac{4\mu\beta Q \cos^2 \pi\alpha_1}{\pi^2 k(1+\kappa)}\int_{-a}^{a}\frac{(t+a)^{\frac{1}{2}+\alpha_1}(a-t)^{\frac{1}{2}-\alpha_1}}{t-x_1}\sin^{-1}\frac{t}{a}\,\mathrm{d}t\right\}. \tag{58}$$

Similar to the previous approach, the range of the dimensionless parameter λ^* for complete contact is found to be $-1.935 < \lambda^* < 1.443$ if a radius of the parabolic-ended punch is assumed to be $R/a = 100$, and the friction coefficient is taken as $\lambda = 0.2$. While separation occurs, the normal stress can be expressed as

$$\sigma_{22} = \frac{1}{(x_1+c_2)^{\frac{1}{2}+\alpha_1}(c_1-x_1)^{\frac{1}{2}-\alpha_1}}\left\{-\frac{Y}{\pi}\cos \pi\alpha_1 + \frac{4\mu \cos \pi\alpha_1}{R(1+\kappa)}\left[x_1^2 + x_1\left\{\alpha_1(c_1+c_1) + \frac{1}{2}(c_2-c_1)\right\} + \frac{\left(\alpha_1^2-\frac{1}{4}\right)}{2}(c_1+c_2)^2\right] - \frac{4\mu\beta Q \cos^2 \pi\alpha_1}{\pi^2 k(1+\kappa)}\int_{-c_2}^{c_1}\frac{(t+c_2)^{\frac{1}{2}+\alpha_1}(c_1-t)^{\frac{1}{2}-\alpha_1}}{t-x_1}\sin^{-1}\frac{2t-c_1+c_2}{c_1+c_2}\mathrm{d}t\right\}. \tag{59}$$

The contact lengths c_1 and c_2 are determined from the following conditions:

$$-\frac{Y}{\pi} + \frac{4\mu}{R(1+\kappa)}\left[c_1^2 + \alpha_1 c_1(c_1+c_2) + \frac{c_1}{2}(c_2-c_1) + \frac{\left(\alpha_1^2-\frac{1}{4}\right)}{2}(c_1+c_2)^2\right] - \lambda^*\frac{Y \cos \pi\alpha_1}{a\pi^2}\int_{-c_2}^{c_1}\frac{(t+c_2)^{\frac{1}{2}+\alpha_1}(c_1-t)^{\frac{1}{2}-\alpha_1}}{t-c_1}\sin^{-1}\frac{2t-c_1+c_2}{c_1+c_2}\,\mathrm{d}t = 0 \tag{60}$$

and

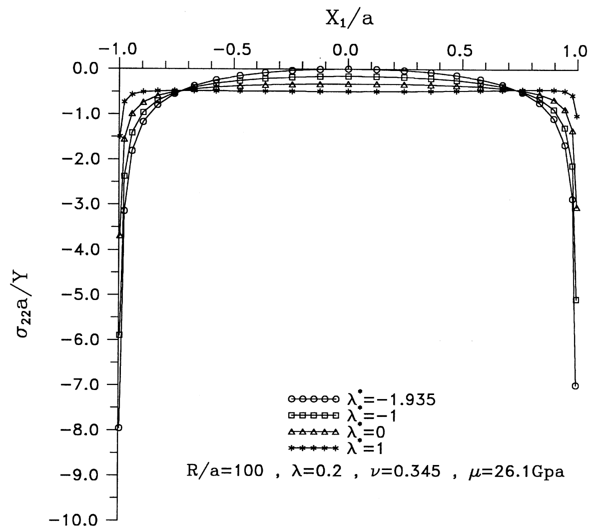

Fig. 14. Contact pressure distribution for perfect contact of frictional parabolic punch.

$$
\begin{aligned}
&-\frac{Y}{\pi}+\frac{4\mu}{R(1+\kappa)}\left[c_2^2-\alpha_1 c_2(c_1+c_2)-\frac{c_2}{2}(c_2-c_1)+\frac{\left(\alpha_1^2-\frac{1}{4}\right)}{2}(c_1+c_2)^2\right] \\
&-\lambda^*\frac{Y\ \cos\ \pi\alpha_1}{a\pi^2}\int_{-c_2}^{c_1}\frac{(t+c_2)^{\frac{1}{2}+\alpha_1}(c_1-t)^{\frac{1}{2}-\alpha_1}}{t+c_2}\sin^{-1}\frac{2t-c_1+c_2}{c_1+c_2} \\
&\mathrm{d}t=0.
\end{aligned}
\tag{61}
$$

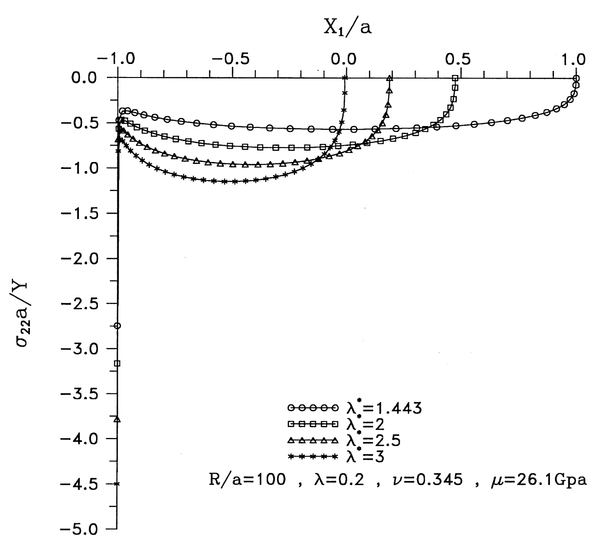

Fig. 15. Contact pressure distribution for separation of frictional parabolic punch.

Typical contact stress distributions for different values of λ^* are displayed in Fig. 14 and Fig. 15 for complete indentation and incomplete indentation, respectively.

6. Concluding remarks

The thermoelastic problem of punch indentation into an elastic half-plane with or without friction is considered in this paper. Based on complex variable theory and the method of analytical continuation, the full field solution of both the temperature and stress functions are obtained analytically. The condition that either perfect contact throughout the punch face or separation occurred at the punch corners is discussed explicitly, which is dependent on the magnitude of resultant force and total heat flux applied over the punch face. The relation of the contact length with the pertinent parameters is found in closed form for frictionless punch problems. For frictional punch problems, both the contact length and the normal stress under the punch must be determined numerically.

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